SUSY QM in Few-body Systems

— Application to the singly heavy-quark Baryons —

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Contents

Central-force Problems in SUSY QM

E.g. The Hydrogen Atom

The Schrödinger eq. is

$$\left[-\boldsymbol{\nabla}^2 - \frac{e^2}{r}\right]\Psi(r,\theta,\varphi) = E\Psi(r,\theta,\varphi)\ .$$

Then we separate the variables as $\Psi(r,\theta,\varphi) = R(r)Y(\theta,\varphi)$, obtaining

$$\begin{cases} \left[-\dfrac{\mathrm{d}^2}{\mathrm{d}r^2} - \dfrac{2}{r}\dfrac{\mathrm{d}}{\mathrm{d}r} + \dfrac{\hat{L}^2}{r^2} - \dfrac{e^2}{r}\right]R(r) = ER(r)\ , \\ \hat{L}^2 Y(\theta,\varphi) = -l(l+1)Y(\theta,\varphi) \end{cases}$$

where $Y(\theta,\varphi)$ is the Spherical Harmonics.

With the substitution $u(r) = rR(r)$, the radial eq. is reduced to

$$\left[-\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{l(l+1)}{r^2} - \frac{e^2}{r}\right] u(r) = Eu(r) \ .$$

One can "factorize" the Hamiltonian and the equation becomes

$$\left(-\frac{\mathrm{d}}{\mathrm{d}r} - \frac{l+1}{r} + \frac{e^2}{2(l+1)}\right)\left(\frac{\mathrm{d}}{\mathrm{d}r} - \frac{l+1}{r} + \frac{e^2}{2(l+1)}\right) u(r) = \left(E + \frac{e^4}{4(l+1)^2}\right) u(r) \ .$$

Therefore, the energy eigenvalues are

$$E_n = -\frac{e^4}{4(n+l+1)^2} \ ,$$

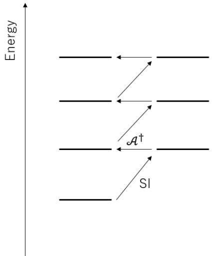
and the wave function for each eigenvalue is also obtained.

$$\boxed{\left(-\frac{\mathrm{d}}{\mathrm{d}r}-\frac{l+1}{r}+\frac{e^2}{2(l+1)}\right)\left(\frac{\mathrm{d}}{\mathrm{d}r}-\frac{l+1}{r}+\frac{e^2}{2(l+1)}\right)u(r)=\left(E+\frac{e^4}{4(l+1)^2}\right)u(r)}$$

- For the ground state,

$$\left(\frac{\mathrm{d}}{\mathrm{d}r}-\frac{l+1}{r}+\frac{e^2}{2(l+1)}\right)u_0(r)=0 \quad\Longrightarrow\quad u_0(r)\propto r^{l+1}\mathrm{e}^{-\frac{r}{2(l+1)}}\,.$$

- For the excited states, the wave functions are obtained by using the following structure of SUSY QM:



How to Deal with Few-body Systems?

In general, for N-body problems, the Hyperspherical formalism is employed. Now we restrict ourselves to the $N = 3$ case.

J. L. Ballot and M. Fabre de la Ripelle, Annals Phys. **127**, 62 (1980)

1. We use the Jacobi relative coordinates:

$$\boldsymbol{\rho}_1 := \mathbf{r}_1 - \mathbf{r}_2 \ , \quad \boldsymbol{\rho}_2 := \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2} - \mathbf{r}_3 \ , \quad \boldsymbol{\rho}_3 := \mathbf{R}_{\mathrm{CM}} \ .$$

2. The kinetic energy operator is written as

$$\hat{T} = -\frac{1}{2m_{\mathrm{tot}}}\boldsymbol{\nabla}^2_{\mathbf{R}_{\mathrm{CM}}} - \frac{1}{2\mu}\left(\boldsymbol{\nabla}^2_{\boldsymbol{\rho}_1} + \boldsymbol{\nabla}^2_{\boldsymbol{\rho}_2}\right) \ .$$

where μ is the reduced mass (We use the natural system of units hereafter). Then the Schrödinger eq. for the internal dynamics is

$$\left[-\frac{1}{2\mu}\left(\boldsymbol{\nabla}^2_{\boldsymbol{\rho}_1} + \boldsymbol{\nabla}^2_{\boldsymbol{\rho}_2}\right) + V\right]\Psi(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2) = E\Psi(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2) \ .$$

3. Here we define hyperradius and hyperangle:

$$x := \sqrt{{\rho_1}^2 + {\rho_2}^2}\ , \quad \xi := \arctan\frac{\rho_1}{\rho_2}\ .$$

4. Especially for the equal mass systems with central forces, one can separate the equation as following:

$$\begin{cases} \left[-\dfrac{\mathrm{d}^2}{\mathrm{d}x^2} - \dfrac{5}{x}\dfrac{\mathrm{d}}{\mathrm{d}x} + \dfrac{\hat{L}^2}{x^2} + 2\mu V(x)\right] X(x) = 2\mu E X(x)\ , \\ \hat{L}^2 \mathscr{Y}(\Omega) = -\gamma(\gamma+4)\mathscr{Y}(\Omega) \end{cases}$$

where Ω denotes the set of five angles $\{\xi, \theta_1, \varphi_1, \theta_2, \varphi_2\}$ and $\mathscr{Y}(\Omega)$ is the Hyperspherical Harmonics:

$$\mathscr{Y}(\Omega) \equiv \mathcal{N}(\sin\xi)^{l_\rho}(\cos\xi)^{l_\lambda} P_n^{l_\rho+1/2, l_\lambda+1/2}(\cos 2\xi) Y_{l_\rho}^{m_\rho}(\theta_\rho, \varphi_\rho) Y_{l_\lambda}^{m_\lambda}(\theta_\lambda, \varphi_\lambda)$$

and

$$\gamma = l_\rho + l_\lambda + 2n\ .$$

The Coordinates Setting

Question: How can we deal with non-equal mass systems?

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We use modified coordinates to describe the positions of the particles:

S. Rosati, World Scientific (2002)

$$\mathbf{x}_i := \sqrt{\frac{m_i}{M}}\ \mathbf{r}_i\ ,$$

instead of the ordinary $\mathbf{r}_i$, where M is the reference mass. Our definitions of Jacobi coordinates here are

$$\boldsymbol{\rho} := \sqrt{\frac{m_2}{m_1+m_2}}\mathbf{x}_1 - \sqrt{\frac{m_1}{m_1+m_2}}\mathbf{x}_2\ ,$$

$$\boldsymbol{\lambda} := \sqrt{\frac{m_3 m_1}{m_{\mathrm{tot}}(m_1+m_2)}}\mathbf{x}_1 + \sqrt{\frac{m_2 m_3}{m_{\mathrm{tot}}(m_1+m_2)}}\mathbf{x}_2 - \sqrt{\frac{m_1+m_2}{m_{\mathrm{tot}}}}\mathbf{x}_3\ ,$$

$$\tilde{\mathbf{R}}_{\mathrm{CM}} := \frac{\sqrt{M}}{m_{\mathrm{tot}}}\left(\sqrt{m_1}\mathbf{x}_1 + \sqrt{m_2}\mathbf{x}_2 + \sqrt{m_3}\mathbf{x}_3\right)\ .$$

The Model

We follow the same procedure as the equal mass case. The Schrödinger eq. is

$$\left[-\frac{1}{2M}\left(\boldsymbol{\nabla}_{\boldsymbol{\rho}}^2+\boldsymbol{\nabla}_{\boldsymbol{\lambda}}^2\right)+V(x)\right]\Psi(x,\Omega)=E\Psi(x,\Omega)$$

with

$$V(x)=ax+bx^2-\frac{c}{x}\ , \quad \text{where } x=\sqrt{\rho^2+\lambda^2}\ .$$

- ax: The linear term
  (This satisfies the experimental data of the ground state mass.)
- bx^2: The correction of two-body forces
  (This allows us to factorize the Hamiltonian.)
- $-\dfrac{c}{x}$: The Coulomb-like term (We fix $c=-\dfrac{2}{3}\alpha_s$.)

Now we take $M = 1\ \mathrm{MeV}$ without loss of generality.

$$\Longrightarrow \begin{cases} \left[-\frac{\mathrm{d}^2}{\mathrm{d}x^2} - \frac{5}{x}\frac{\mathrm{d}}{\mathrm{d}x} + \frac{\hat{L}^2}{x^2} + 2\left(ax + bx^2 - \frac{c}{x}\right)\right] X(x) = 2EX(x) \\ \\ \hat{L}^2\mathscr{Y}(\Omega) = -\gamma(\gamma+4)\mathscr{Y}(\Omega) \end{cases}$$

Substituting $\chi(x) = x^{5/2}X(x)$, we get the radial eq.:

$$\left[-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{-\gamma(\gamma+4) + \frac{15}{4}}{x^2} + 2\left(ax + bx^2 - \frac{c}{x}\right)\right]\chi(x) = 2E\chi(x) \ ,$$

whose Hamiltonian can be factorized and the equation becomes

$$\left(-\frac{\mathrm{d}}{\mathrm{d}x} + Ax - \frac{B}{x} + D\right)\left(\frac{\mathrm{d}}{\mathrm{d}x} + Ax - \frac{B}{x} + D\right)\chi(x) = (2E + D^2 - 2AB - A)\chi(x) \ .$$

Therefore, the energy eigenvalues are

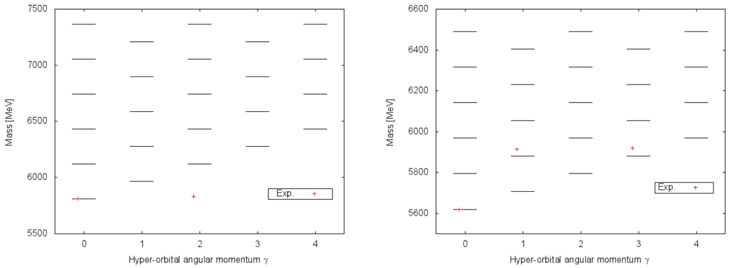
$$2E_n = -\frac{c^2}{4(\gamma + n + \frac{5}{2})^2} + 2\sqrt{b}\left(\gamma + \frac{5}{2}\right) + (4n+1)\sqrt{b} \ .$$

The Mass Spectra of (qq'b)

The mass is calculated as

$$M_n = \sum_{i=1}^{3} m_i + E_n \ .$$

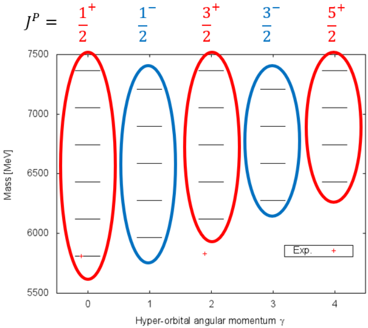
Here are the mass spectra of Σ_{b}^{+}(uub) and Λ_{b}^{0}(udb):



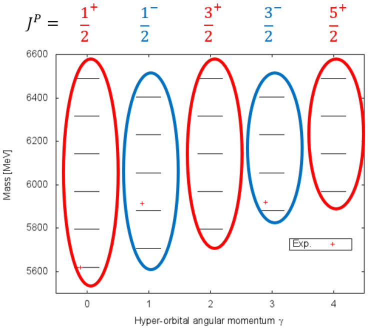
(a) Σ_b^+(uub) (b) Λ_b^0(udb)

Figure 1: The mass spectra

The Mass Spectra of (qq'b)



(a) Σ_b^+(uub)



(b) Λ_b^0(udb)

Figure 2: The mass spectra

The Wave Functions of (uub)

The wave functions $\Psi(x,\Omega) = x^{-5/2}\chi(x)\mathscr{Y}(\Omega)$ are also obtained analytically in our scheme. Here, we define the existence probability as

$$P(x,\xi) = \int \mathrm{d}\Omega_\rho \int \mathrm{d}\Omega_\lambda \, |\Psi(x,\Omega)|^2 \ .$$

We plot it in the $(r_{\text{uu}}, r_{\text{uu-b}})$-plane:

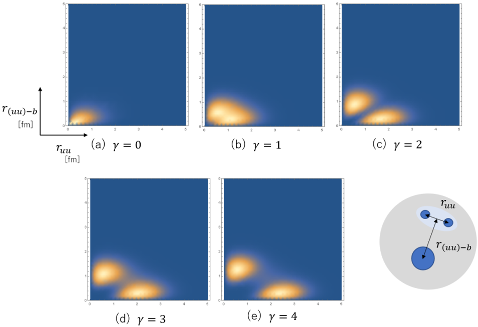


Figure 3: The density plots of P

- $x = \sqrt{\rho^2 + \lambda^2}$
- $\xi = \arctan \dfrac{\rho}{\lambda}$
- $r_{\text{uu}} \propto \rho$
- $r_{\text{uu-b}} \propto \lambda$

Baryonic Structure in terms of Our Model

- As the hyper-angular momentum γ grows, two different modes of internal dynamics arise.

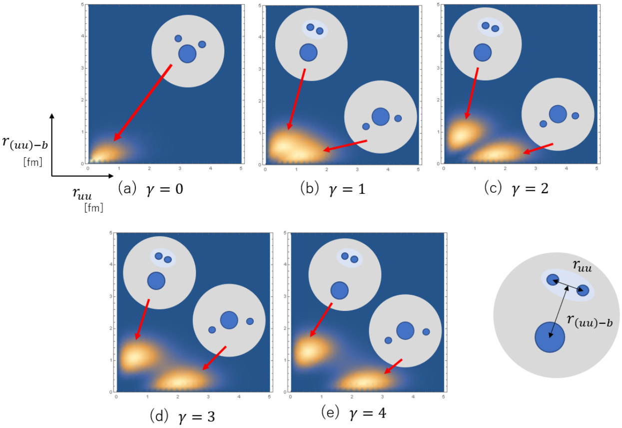


Figure 4: The internal structure of the baryon

Conclusion and Future Works

- We can solve analytically non-equal mass problems in terms of SUSY QM.
  - We find that, in the Hyperspherical formalism, the Hamiltonian of hyperradial equations can be factorized with carefully chosen potentials (for any N).
  - The tricky way of choosing coordinates allows us to generalize the usual (equal mass) Hypersherical formalism.
- As a result, we find the following features of the structure of the baryonic systems.
  - We obtain the mass spectra of the baryons.
  - The baryons have two modes of internal dynamics.
- Is it possible to construct the Hamiltonian $\mathcal{H} = \begin{pmatrix} H_{\mathrm{M}} & 0 \\ 0 & H_{\mathrm{B}} \end{pmatrix}$ where H_{M} and H_{B} are SUSY partners?
  - Is there a meson-baryon symmetry with the meson-baryon supermultiplet $\begin{pmatrix} \phi_{\mathrm{M}} \\ \phi_{\mathrm{B}} \end{pmatrix}$?

— Three quarks for Muster Mark! (James Joyce, *Finnegans Wake*)

Thank you for listening.